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The Complete Reducibility of Some $GF(2)A_7$-Modules

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Abstract—It is proved that, if G is a finite group with a nontrivial normal 2-subgroup Q such that $G/Q \cong A_7$ and an element of order 5 from G acts freely on Q, then the extension G over Q is splittable, Q is an elementary abelian group, and Q is the direct product of minimal normal subgroups of G each of which is isomorphic, as a G/Q-module, to one of the two 4-dimensional irreducible $GF(2)A_7$-modules that are conjugate with respect to an outer automorphism of the group A_7.

Keywords: finite group, $GF(2)A_7$-module, completely reducible representation, prime graph.

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INTRODUCTION

Many researchers are interested in various problems of recognition, i.e., of characterization of a group according to some collection of its arithmetic parameters. One of such problems is the characterization of a finite group by its prime graph.

Let G be a finite group. Denote by $\pi(G)$ the set of prime divisors of the order of G. The *prime graph* $\Gamma(G)$ of the group G is the graph with vertex set $\pi(G)$ such that two different vertices p and q are adjacent if and only if G contains an element of order pq. Denote the number of connected components of $\Gamma(G)$ by $s(G)$ and the set of connected components by $\{\pi_i(G) \mid 1 \le i \le s(G)\}$. For a group G of even order, we assume that $2 \in \pi_1(G)$.

Within the general problem of investigating finite groups by the properties of their prime graphs, the class of groups with disconnected prime graph is of particular interest since this class is a wide generalization of the class of finite Frobenius groups. This is seen immediately from the known structural Gruenberg–Kegel theorem [20, Theorem A] on finite groups with disconnected prime graph. The role of Frobenius groups in the finite group theory is exceptional.

The investigation of groups with disconnected prime graph involves nontrivial problems connected with modular representations of finite groups. Let us consider such a problem. Let G be a finite group with disconnected prime graph isomorphic neither to a Frobenius group nor to a 2-Frobenius group. Denote by $F(G)$ the Fitting subgroup of G. Then, by the Gruenberg–Kegel theorem, the group $\overline{G} = G/F(G)$ is almost simple and is known in view of results from [2, 13, 20]. Assume that $F(G) \neq 1$. Any connected component $\pi_i(G)$ of the graph $\Gamma(G)$ for $i > 1$ corresponds to a nilpotent isolated $\pi_i(G)$-Hall subgroup $X_i(G)$ of G. Any nontrivial element x from $X_i(G)$ $(i > 1)$

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acts fixed-point-freely (freely) on $F(G)$; i.e., $C_{F(G)}(x) = 1$. Let K and L be two neighboring terms of a chief series of G and, moreover, $K < L \leq F(G)$. Then, the (chief) factor $V = L/K$ is an elementary abelian p-group for some prime p. It is called a *p-chief factor* of G and can be considered as a faithful irreducible $GF(p)\overline{G}$-module (since $C_{G/K}(V) = F(G)/K$); moreover, any nontrivial element from $X_i(G)$ $(i > 1)$ acts freely on V. Therefore, the problem of investigating the structure of the group G reduces largely to the problem of describing irreducible $GF(p)\overline{G}$-modules on which some element of prime order (different from p) from $\overline{G}$ acts freely; this problem is also of independent interest.

Refining this problem, we come to the following statement. Suppose that G is a finite group, Q is a nontrivial normal subgroup in G, $\overline{G} := G/Q$ is a known group, and an element of some prime order from $G \setminus Q$ acts on Q freely. By Thompson's theorem [19], the subgroup Q is nilpotent. The following questions are now natural.

(1) *What are the chief factors of the group G in Q?*

(2) *What is the structure of the group Q?*

(3) *Is the action of $\overline{G}$ on Q completely reducible if Q is elementary abelian*?

(4) *Is the extension G over Q splittable*?

Despite the importance of these problems, there are not so many results in this area. The first investigation devoted to the case when $\overline{G}$ is a simple nonabelian group was Higman's classical paper [10]. For the case when the group $\overline{G}$ is isomorphic to $L_2(2^m)$, $m \geq 2$, and an element of order 3 from G acts on Q freely, Higman gave complete (positive) answers to all the questions formulated above. In particular, Q is an elementary abelian 2-group, the action of $\overline{G}$ on Q is completely reducible, and any 2-chief factor of the group G is isomorphic to the natural $GF(2^m)SL_2(2^m)$module. Later, Martineau [14,15] obtained a similar result for the case when $\overline{G}$ is isomorphic to $Sz(2^n)$ and an element of order 5 from G acts on Q freely. Continuing Higman's paper, Stewart [18] showed that $Q = 1$ in the case when the group $\overline{G}$ is isomorphic to $L_2(q)$, q is odd, $q > 5$, and an element of order 3 from G acts on Q freely. Papers by Prince [16], Zurek [21], and Holt and Plesken [11] were devoted to the case when $Q = O_2(G)$, the group $\overline{G}$ is isomorphic to A_5, and an element of order 5 from G acts on Q freely. This case turned out to be difficult since Q can be nonabelian. Prince [16] and Zurek [21] gave complete (positive) answers to questions (1), (3), and (4). In particular, Q is the product of $\overline{G}$-invariant subgroups Q_i isomorphic to either a homocyclic 2-group of rank 4 or a special 2-group of order 2^8 with center of order 2^4 (which is isomorphic to the unipotent radical of some parabolic maximal subgroup in $U_5(2)$). Moreover, in the first case, any 2-chief factor of G lying in Q_i is isomorphic to the orthogonal (permutation) $GF(2)A_5$-module, and, in the second case, the group $Z(Q_i)$ is isomorphic to the orthogonal $GF(2)A_5$-module and $Q_i/Z(Q_i)$ is isomorphic to the natural $GF(4)SL_2(4)$-module. According to Higman's earlier result, a theoretical upper estimate for the nilpotency class of Q is 6. In [21], Zurek supposed that this estimate is 2. However, later, Holt and Plesken [11] proved that the nilpotency class of Q does not exceed 3 and constructed an example of the group Q of order 2^{28} where this estimate is attained. Using a computer program, they also showed the absence of examples of smaller orders. Prince [16, 17] showed that, in the case when $Q = O_2(G)$, the group $\overline{G}$ is isomorphic to A_6, and an element of order 5 from G acts on Q freely, questions (1)–(4) have positive solutions. In [1,3,4], it is proved that, in the case when $O(G) \neq 1$, the group $\overline{G}$ is isomorphic to A_6, and an element of order 5 from G acts on Q freely, the group $O(Q)$ is abelian, $O(Q) = O_3(G)$, and 3-chief factors of G are isomorphic to a 4-dimensional irreducible permutation $GF(3)\overline{G}$-module.

If $Soc(\overline{G})$ is a group of Lie type over a field of characteristic p, then, for solving problem (1), we can apply Guralnick and Tiep's result [9], which describes all unisingular simple groups of Lie type. A simple group S of Lie type over a field of characteristic p is called *unisingular* if any element $s \in S$ has a nontrivial fixed point in any nontrivial abelian p-group on which S acts. If the table of irreducible p-modular Brauer characters for the socle of the group $\overline{G}$ is known, then, for solving problem (1), we can apply Lemma 1 formulated below.

In the present paper, we prove the following theorem.

Theorem. *Let G be a finite group with a nontrivial normal 2-subgroup Q such that $G/Q \cong A_7$. Assume that an element of order 5 from G acts freely on Q. Then, the extension G over Q is splittable, Q is elementary abelian, and Q is the direct product of minimal normal subgroups of G each of which is isomorphic, as a $GF(2)G/Q$-module, to one of the two 4-dimensional irreducible $GF(2)A_7$-modules that are conjugate with respect to an outer automorphism of the group A_7.*

1. NOTATION AND AUXILIARY RESULTS

Our notation and terminology are mostly standard and can be found in [5–8, 12].

The following useful result is well known (see, for example, [1, Lemma 4]).

Lemma 1. *Suppose that G is a finite simple group, F is a field of characteristic $p > 0$, V is an absolutely irreducible FG-module, and β is a Brauer character of V. If g is an element from G of a prime order different from p, then*

$$\dim C_V(g) = (\beta|_{\langle g\rangle}, 1|_{\langle g\rangle}) = \frac{1}{|g|}\sum_{x\in\langle g\rangle}\beta(x).$$

We will also need the following lemma.

Lemma 2. *Let G be a group isomorphic to A_7, and let V be a 4-dimensional irreducible $GF(2)G$-module. If $S \in Syl_2(G)$, then $\dim C_V(S) = 1$.*

Proof. In view of [8], all involutions in G are conjugate. The group G contains a subgroup D isomorphic to D_{10}. The subgroup D is generated by two involutions t_1 and t_2. It is clear that $\dim C_V(t_1) = \dim C_V(t_2) \in \{2, 3\}$. If $\dim C_V(t_1) = 3$, then $C_V(D) \neq 0$, a contradiction with the fact that an element of order 5 from D acts on V freely. Hence, $\dim C_V(t) = 2$ for all involutions t from G. Suppose that $S \in Syl_2(G)$ and $W = C_V(S)$. It is clear that $W \neq 0$. Assume that $\dim W > 1$. Then, $\dim W = 2$. The subgroup S is contained in a subgroup $R \cong S_4$ from G. Let $U = O_2(R)$. Then, the subspace $W = C_V(U)$ is R-admissible and, consequently, R centralizes W. The subgroup R contains a subgroup $\langle x\rangle \rtimes \langle t\rangle$ isomorphic to S_3. Then, $V = [\langle x\rangle, V] \oplus W$, and the 2-dimensional subspace $[\langle x\rangle, V]$ is t-admissible. Therefore, $C_{[\langle x\rangle,V]}(t) \neq 0$, and, consequently, $\dim C_V(t) = 3$, a contradiction. The lemma is proved.

2. THE PROOF OF THE THEOREM

Suppose that the conditions of the theorem are fulfilled. Define $\overline{G} = G/Q$. Since G contains a subgroup isomorphic to A_6, the subgroup Q is an elementary abelian 2-group in view of [16, Theorem 2]; hence, we can consider it as a $GF(2)\overline{G}$-module. By [6], any 2-chief factor of G is isomorphic to one of the two irreducible 4-dimensional $GF(2)A_7$-modules that are conjugate with respect to an outer automorphism of the group A_7. Using induction on the number of 2-chief factors of the group G, it is sufficient to consider the case when G has exactly two 2-chief factors. In this case, $|Q| = 2^8$. Let Q_1 be a proper G-admissible subspace from Q. Then, $Q_1 \trianglelefteq G$ and

$|Q_1| = 2^4$. It follows from the condition that $\overline{G}$ acts irreducibly on Q_1 and Q/Q_1 (since a Sylow 5-subgroup from $\overline{G}$ acts so). The group $\overline{G}$ contains a subgroup $\overline{M}$ isomorphic to A_6. Let M be a complete preimage of $\overline{M}$ in G. By [17, Theorem 1], Q contains a subgroup Q_2 of order 16 such that $Q = Q_1 \times Q_2$ and M normalizes Q_2. In view of [6] and Lemma 1, $\overline{M}$ contains an element of order 3 acting freely on Q_1 and Q/Q_1. Therefore, $\overline{M}$ acts transitively on nontrivial elements of each of the groups Q_1 and Q/Q_1. Assume that t is an involution from Q_2, $T = Q_1 \times \langle t\rangle$, and $N = N_G(T)$. Then, $|G\colon N| = 15$. In view of [8], we have $\overline{N} \cong L_2(7)$ and $\overline{N_M(T)} \cong S_4$.

Take in N an element z of order 7. It is clear that $|C_Q(z)| = 4$. Further, by Maschke's theorem, $C_Q(z) = \langle v\rangle \times \langle t_1\rangle$, where $Q_1 = [Q_1, \langle z\rangle] \times \langle v\rangle$ and $T = Q_1 \times \langle t_1\rangle$. Therefore, $\langle z\rangle$-orbits on T are exhausted by four one-element orbits $(\{1\}, \{v\}, \{t_1\}, \{vt_1\})$ and four orbits of length 7. If $t \in \{t_1, vt_1\}$, then $|G : C_G(t)| = 15$ and, consequently, $Q_2 = \{1\} \bigcup t^G \lhd G$; i.e., G acts completely reducibly on Q. Therefore, we assume in the sequel that $|G : C_G(t)| = 15 \times 7 = 105$.

Since Q_2 and T are normal in $N_M(T)$, the subgroup $\langle t\rangle = Q_2 \bigcap T$ is normal in $N_M(T)$; i.e., $C_G(t) = N_M(T)$.

We have $N_N(z) = (\langle v\rangle \times \langle t_1\rangle) \times (\langle z\rangle \leftthreetimes \langle x\rangle)$, where $\langle z, x\rangle$ is a Frobenius group of order 21. If $C_N(\langle v, t_1\rangle) > N_N(\langle z\rangle)$, then $\langle v, t_1\rangle$ centralizes N. Then, however, the subgroup $\langle v\rangle \times \langle t_1\rangle \times \langle t\rangle$ centralizes $N_M(T)$ and, in particular, centralizes a Sylow 2-subgroup in $\overline{M}$, which contradicts Lemma 2.

Thus, $C_N(\langle v, t_1\rangle) = N_N(\langle z\rangle)$.

Assume that v centralizes N. Then, $C_N(t_1) = C_N(vt_1) = N_N(\langle z\rangle)$ and, hence, $|N\colon C_N(t_1)| = |N\colon C_N(vt_1)| = 8$.

Let us show that t_1 and vt_1 are not conjugate in N. Assume by contradiction that there exists an element $g \in N$ such that $gt_1g^{-1} = vt_1$. Then, g does not lie in $C_N(t_1)$. Since $\overline{N} = \overline{N_N(\langle z\rangle)S}$, where S is a Sylow 2-subgroup from $N_M(T)$, we can assume that $g \in S$. We have $g^2t_1g^{-2} = t_1$; i.e., g^2 centralizes the subgroup $\langle v, t_1\rangle$ and, hence, $\overline{N_N(\langle v, t_1\rangle)}$ contains $\langle\overline{N_N(\langle z\rangle)}, \overline{g}\rangle = \overline{N}$. Consequently, $\langle v, t_1\rangle \lhd N$ and $|N\colon C_N(t_1)| = 2$, which contradicts the equality proved above.

Thus, $T \setminus Q_1$ can be decomposed into N-classes of conjugate elements with representatives t, t_1, and vt_1, which contradicts the equality $|T \setminus Q_1| = 16$.

Thus, v does not centralize N and, hence, $C_N(v) = N_N(\langle z\rangle)$, which implies $|N : C_N(v)| = 8$. We can assume that $|N : C_N(t_1)| = 8$.

We have $T \setminus Q_1 = t^N \bigcup t_1^N \bigcup \{vt_1\}$. Therefore, $\langle t\rangle \times \langle vt_1\rangle$ centralizes $N_M(T)$ and, consequently, centralizes in $\overline{G}$ the subgroup $\overline{N_M(T)}$, which is isomorphic to S_4. Hence, the quadratic subgroup $\langle tQ_1\rangle \times \langle t_1Q_1\rangle$ from Q/Q_1 centralizes a Sylow 2-subgroup from $\overline{N_M(T)}$, which is a Sylow 2-subgroup in $\overline{G}$; this contradicts Lemma 2.

The complete reducibility is proved.

Let us prove now that the extension G over Q is splittable. By [16, Theorem 2], the extension of Q by $\overline{M}$ is splittable. Now, Gaschütz's theorem [7, (10.4)] implies that G is splittable over Q.

The theorem is proved.

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